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"Application of the Wang Programmable Calculator to the Burmester Problem for Finite Displacement Synthesis of Planar Mechanisms"

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Preface

This project was initiated and directed by Dr. F. R. E. Crossley from Aug. 10, 1966 until Dec. 31, 1968 when he left Georgia Tech.

A new phase was conducted under the sole direction of Dr. J. R. Baumgarten until June 30, 1970 when he left Georgia Tech.

Dr. W. M. Williams had previously been named co-principal investigator on a contingent basis and assumed responbilility in Sept. 1970. On April 1, 1971 he was first able to obtain a qualified Graduate Assistant in the person of Mr. Dennis P. Carlson and to proceed with the work reported on here. After 3 months and a total expenditure of $1209, the project expired on June 30, 1970 as previously negotiated by Dr. Baumgarten.

Final refusal to extend the project was received from NSF on Sept. 21, 1971.

The original proposal of Dr. Crossley was directed to the structural analysis and type synthesis of mechanisms in both two and three dimensions as a means of advancing the methodology of Mechanism Design.

The activities of Dr. Baumgarten were in the area of machine dynamics and the damping effects of impacting bodies, however, his brief summary did propose "to continue some level of investigation in the field of three dimensional kinematic synthesis as described in the original proposal".

It was this latter intent involving procedures and design methodology for synthesis of mechanisms which was elected to be carried out in the final phase by Dr. Williams and Mr. Carlson. As a first step the use of the Wang 720 Programmable Calculator was investigated as an optimum means to permit the designer of planar mechanisms to employ the sophistications of Burmester Theory without the cost of conventional computer programming. Investigation in three dimensions could not be initiated during the curtailed life of the project.

I. Introduction

The purpose of this work is to make available the solution of the two dimensional Burmester problem to those who have a Wang 720 Advanced Programmable Calculator and to demonstrate the effectiveness of this type of calculator in solving mechanism synthesis problems.

Burmester theory identifies those points on a body permitting certain sequential displacements. The solution to the two dimensional Burmester problem is well known having been solved graphically by Burmester and analytically by others using either complex algebra or the algebra of real numbers. The solution would be prohibitive to designers if done manually and the programming or computer capability for a large computer may not always be at hand. For these reasons, the small computers which are relatively inexpensive and easy to program, may find valuable uses in the application of kinematic theory to design practices.

II. The Burmester Problem

When designing a linkage that will carry a body through sequential positions, it is most convenient to find points on the body whose successive positions lie on the circumference of a circle. The significance of the circle is that it corresponds to the path generated by a point on a link that is connected by a simple pivot or revolute joint. By selecting two such points it is possible to design a four bar linkage with revolute couples that will carry a body through the desired positions.

If three positions are specified for the body, the problem is simple as it takes only three non-colinear points to define a circle in the plane. The three successive positions of any two points would define unique circles, the centers of which would locate fixed pivots. Each pivot would be connected with its corresponding point by a link whose length is equal to the radius of the circle. The resulting four bar linkage would have the body as its coupler.

If four exact positions are specified for the body, the problem becomes more difficult and a graphical solution is employed. The body contains a locus of points having all four positions on the circumference of some circle. This locus is a line on the body and is generally called the circle point curve. The centers of the circles fall on another line which is called a center point curve. For each center point there exists a unique circle point. Since there are an infinite number of circle point - center point pairs, there are an infinite number of solutions and the solutions can be employed to construct four bar linkages as in the case with three accuracy points.

If the body is required to move through five exact positions, Burmester theory predicts that there are a maximum of four points in the body whose five sequential positions define circles. The input for the Burmester problem is the five desired positions of the moving reference frame (the body) -- each position specified by two coordinates and an angle. The problem is solved when the four Burmester points are found along with the sizes and centers of their corresponding circles.

Formulation and Derivation

The five positions of a body relative to a fixed frame are specified by a_i, b_i, θ_i for i = 1, 2, 3, 4, 5 (see Fig. 1). It is then desired to determine the (x,y) points on the body which lie on circles in the fixed plane and to identify each circle by radius (R) and center coordinates (D,E).

The equation of the circular path in the fixed frame is

$$(X - D)^2 + (Y - E)^2 = R^2. \tag{1}$$

Substituting the following transformation into equation (1)

$$X = a + x \cos\theta - y \sin\theta$$

$$Y = b + y \cos\theta + x \sin\theta$$

yields

$$(a - y \sin\theta + \cos\theta - D)^2 + (b + x \sin\theta + y \cos\theta - E)^2 = R^2$$

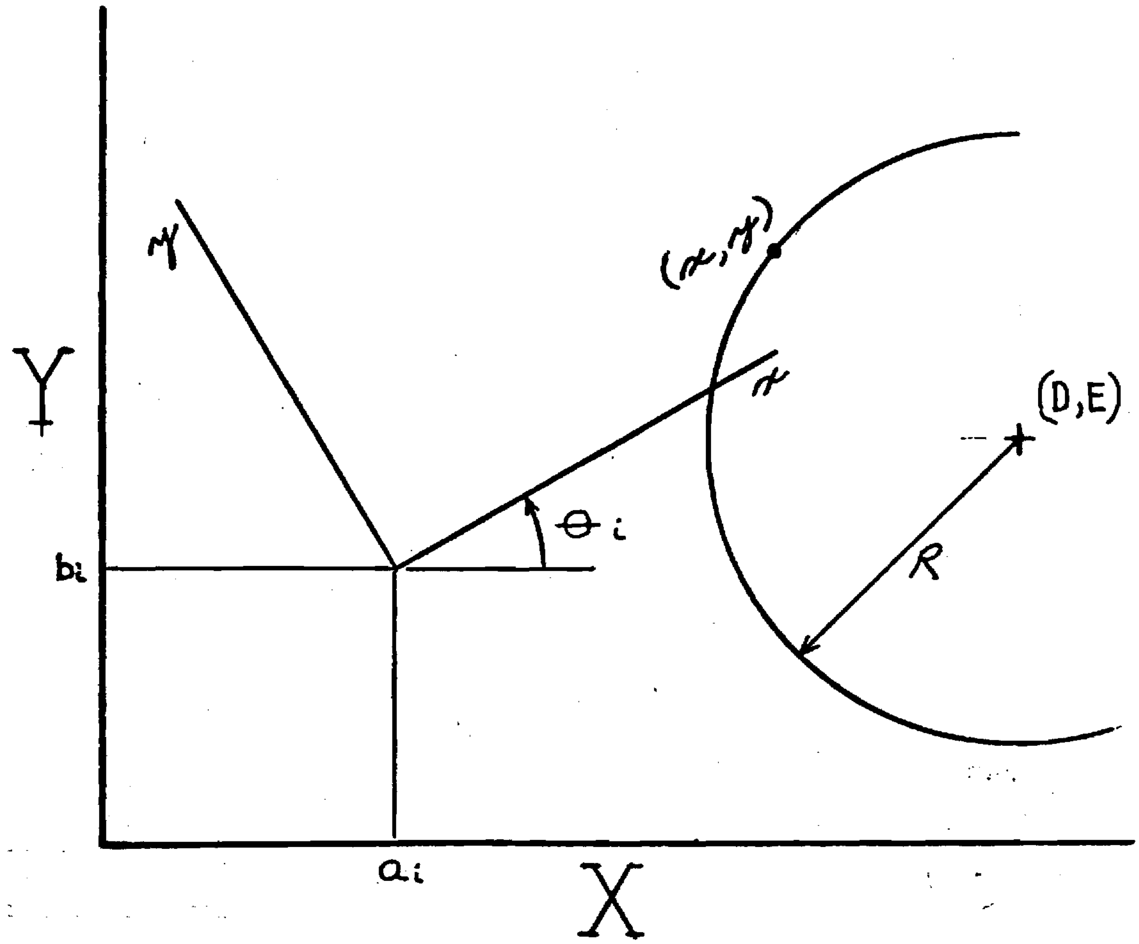

Figure 1. COORDINATE SYSTEMS AND PARAMETERS IN THE TWO-DIMENSIONAL BURMESTER PROBLEM.

Expanding and rearranging

$$[x^2(\sin^2\theta + \cos^2\theta) + y^2 \ (\sin^2\theta + \cos^2\theta) + X^2 + Y^2 + R^2] \qquad (2)$$

$$+ [yD - xE] \ 2 \sin\theta - [yE + X \ D] \ 2 \cos\theta$$

$$+ [x] \ 2r \sin (\gamma + \theta) + [y] \ 2r \cos(\gamma + \theta) - [D] \ 2a$$

$$- [E] \ 2b = - (a^2 + b^2)$$

where $r = \sqrt{a^2 + b^2}$

$$\gamma = \tan^{-1} \frac{a}{b}$$

In order to emphasize the linearity of equation (2) define the following terms:

$$\begin{aligned}
K_1 &\equiv x^2 + y^2 + D^2 + E^2 - R^2 \\
K_2 &\equiv y \ D - x \ E \\
K_3 &\equiv y \ E + x \ D \\
K_4 &\equiv y \\
K_5 &\equiv x \\
K_6 &\equiv D \\
K_7 &\equiv E
\end{aligned} \qquad (3)$$

Substituting these values equation (2) becomes:

$$K_1 + K_2 \ 2 \sin\theta - K_3 \ 2 \cos\theta + K_4 \ 2r \cos(\theta + \gamma)$$

$$+ K_5 \ 2r \sin(\theta + \gamma) - K_6 \ 2a - K_7 \ 2b = - (a^2 + b^2) \ . \qquad (4)$$

Define

$$K_1 = c_1 + d_1K_6 + e_1 K_7$$
$$K_2 = c_2 + d_2K_6 + e_2 K_7$$
$$K_3 = c_3 + d_3K_6 + e_3 K_7 \qquad (5)$$
$$K_4 = c_4 + d_4K_6 + e_4 K_7$$
$$K_5 = c_5 + d_5K_6 + e_5 K_7$$

where the c's, d's, and e's are numerical constants

Substitute (5) into (4) and equate the coefficients of the like terms to obtain three separate equations:

$$c_1 + c_2 (2 \sin\theta) - c_3 (2 \cos\theta) + c_4[2r \cos(\theta + \gamma)] + c_5 [2r \sin(\theta+ \gamma)] = - (a^2 + b^2) \qquad (5c)$$

$$d_1 + d_2 (2 \sin\theta) - d_3 (2 \cos\theta) = d_4 [2r \cos(\theta + \gamma)] + d_5 [2r \sin(\theta + \gamma)] = 2a \qquad (5d)$$

$$e_1 + e_2 (2 \sin\theta) - e_3 (2 \cos\theta) + e_4 [2r \cos(\theta + \gamma)] + e_5 [2r \sin(\theta + \gamma)] = 2b \qquad (5e)$$

Each equation, 5c, 5d, 5e, can be written five times for the five positions a_i, b_i, θ_i, $i = 1, 2, 3, 4, 5$. These three sets of five simultaneous equations can be solved to yield the 15 constants c_i, d_i, e_i, $i = 1,2,3,4,5$.

K_1, K_2, K_3, K_4, & K_5 can then be expressed as functions of K_6 and K_7 and numerical constants from equations (5).

Selecting the second and third expressions from (3) as the compatability equations, and writing in terms of the K's:

$$K_2 = K_4 K_6 - K_5 K_7$$
$$K_3 = K_5 K_6 + K_4 K_7 \; . \qquad (6)$$

Substituting (5) into (6) and rearranging yields:

$$c_1 + (d_2 - c_4) K_6 + (e_2 + c_5) K_7 + (d_5 - e_4) K_6 K_7 - d_4 K_6^2 + e_5 K_7^2 = 0 \qquad (7)$$

$$e_3 + (d_3 - c_5) K_6 + (e_3 - c_4) K_7 - (d_4 + e_5) K_6 K_7 - d_5 K_6^2 - e_4 K_7^2 = 0 \qquad (8)$$

which are two equations in two unknowns, K_6 and K_7.

By multiplying (7) by e_4 and (8) by e_5 and adding we eliminate the K_7^2 term. The resulting equation can be solved for K_7:

$$K_7 = \frac{(e_4d_4 + e_5d_5) K_6^2 - (e_4d_2 - e_4c_4 + e_5d_3 - e_5c_5) K_6 - e_4c_2 - e_5c_3}{(e_4d_5 - e_4^2 - e_5d_4 - e_5^2) K_6 + (e_4e_2 + e_4c_5 + e_3e_5 - e_5c_4)}$$

To simplify this expression, define:

$$P = e_4d_4 + e_5d_5$$

$$Q = -(e_4d_2 - e_4c_4 + e_5d_3 - e_5c_5)$$

$$S = -(e_4c_2 + e_5c_3)$$

$$T = (e_4d_5 - e_4^2 - e_5d_4 - e_5^2)$$

$$U = (e_43_2 + e_4c_5 + e_3e_5 - e_5c_4) \; .$$

Then

$$K_7 = \frac{P\,K_6^{\,2} + Q\,K_6 + S}{T\,K_6 + U} \quad . \tag{9}$$

Substitute (9) into (8) in order to obtain the expression in K_6:

$$c_3 + L\,K_6 - d_5\,K_6^{\,2} - e_4\left[\frac{PK_6^{\,2} + QK_6 + S}{T\,K_6 + U}\right]^2 +$$

$$\left[\frac{P\,K_6^{\,2} + Q\,K_6 + S}{T\,K_6 + U}\right](M + N\,K_6) = 0$$

where

$$L = (d_3 - c_5)$$

$$M = (e_3 - c_4)$$

$$N = -(d_4 + e_5) \quad .$$

Clearing, the numerator may be set equal to zero yielding:

$$\begin{aligned}
&[K_6^{\,4}\,(-d_5T^2) + K_6^{\,3}\,(T^2\,L - 2T\,Ud_5) + K_6^{\,2}\,(c_3T^2 + 2\,TUL - d_5U^2) \\
&+ K_6\,(2\,TU\,c_3 + L\,U^2) + U^2\,c_3 - (e_4\,P^2)\,K_6^{\,4} - (e_4\,2\,PQ)\,K_6^{\,3} \\
&- e_4\,(2\,PS + Q)\,K_6^{\,2} - (e_4\,2\,QS)\,K_6 - S^2\,e_4 + (PNT)\,K_6^{\,4} \\
&+ K_6^{\,3}\,(PTM + PNU + QNT) + K_6^{\,2}\,(PMN + QTM + QNU + SNT) \\
&+ K_6\,(QUM + SMT + SNU) + SMU] = 0
\end{aligned}$$

Combining coefficients of like terms, the fourth order polynomial in K_6 is obtained :

$$\begin{aligned}
&K_6^{\,4}\ [-d_5T^2 - e_4P^2 + PNT] + \\
&K_6^{\,3}\ [T^2L - 2TUd_5 - e_4 2PQ + PTM + PNU + QNT] + \\
&K_6^{\,2}\ [c_3T^2 + 2TUL - d_5U^2 - e_4\ 2PS - e_4Q^2 + PMU \\
&+ QTM + QNU + SNT] + K_6\ [2\ TU\ c_3 + LU^2 \\
&- e_4\ 2QS + QUM + SMT + SNU] \\
&+ [U^2c_3 - S^2e_4 + SMU] \ = \ 0\ .
\end{aligned} \tag{10}$$

The procedure is to solve equation (10) for $K_6(=D)$, then use equation (9) to solve for $K_7(=E)$. Determine $K_4(=y)$ and $K_5(=x)$ from equation (5). Then use equation (1) to solve for R.

III. Applying the Wang Programmable Calculator

A. Program Description

This program consists of one minor and five major program blocks stored successively on a cassette tape cartridge. Once the first part is loaded by the operator into the machine core and the input data is fed in, the program loads and executes each successive block until all the blocks have been used. All of the data, including the results from the intermediate blocks, is stored in the core. When the complete program has been executed, the final answers are automatically displayed in succession at the operator's command.

The first block, the minor block, is a loading subroutine. The end of each major program calls this subroutine which executes a "load program" operation. This loads the next major block into the core from the tape.

The first major program block takes the five sets of three input variables, a,b,θ, and converts them into the coefficients of the three sets of five simultaneous algebraic equations (5c,d,e). The block uses a portion of the Wang "Trig Pac" program to compute sines, cosines, and inverse tangents. A test has been added that determines the quadrant of the arctangent. The program uses the indirect addressing feature of the Wang to subscript the variables. Since all five sets of a,b, & θ have the same operations performed on them, it was desirable in terms of core space and efficiency to have a loop routine that would perform these operations on all five sets of data. This was done by the indirect addressing technique where the storage address is treated as a variable and increased one unit for each cycle through the loop. A test is made within the loop to determine if five loops have been completed. If less than five loops have been completed, the storage locations are increased by one and the loop is executed again. If five have been completed the loading subroutine is called which loads the second major block.

The second major block of the program solves three sets of five simultaneous algebraic equations(5c,d,e) and stores the results for block three. The standard Wang program for solving linear simultaneous algebraic equations was modified to operate automatically as a subroutine. The loading and displaying routines of the Wang program were removed. A routine to call the coefficients from storage and a routine to store the answers were substituted. Most of the program dealt with loading each set of equations for the solving routine and the looping thru the three similar but different problems. Again a test was included which determined if all the desired loops had been completed.

The third major block takes the fifteen solutions from the second block and calculates the coefficients of the fourth order polynomial (Eq. 10). This block is a straight-forward but cumbersome algebraic manipulation.

The fourth of the major blocks solves for the four roots of the polynomial. The block uses the standard Wang program for the solution of second, third, and fourth order polynomials. The program was modified to make its operation completely automatic. A program that would solve any order polynomial by the half interval search method could be substituted should the need arise in other problems of this type. This half interval search program requires some manual operation, as the limits of the search and size of the step must be supplied by the user according to the particular nature of the problem. The program that was used solves for both real and imaginary roots and both are stored for the fifth block although only the real roots produce valid solutions. Thus the user should check the storage locations of the imaginary parts. A nonzero value at these locations suggests that the root is imaginary and the corresponding solution is invalid.

The fifth block takes the roots of the polynomial and evaluates the five parameters of each solution (x,y,D,E and R).

B. Operating Procedure

Insert Tape; Press: "Tape Rewind";

Press: "Tape Ready";

Key: "Prime"; "Load Program";

Key: a_1 in Y Register,

b_1 in X Register;

Key: "Search, 0";

Key: a_2 in Y,

b_2 in X; "Go";

a_3 in Y,

b_3 in X; "Go";

a_4 in Y;

b_4 in X; "Go";

a_5 in Y;

b_5 in X; "Go";

θ_1 in X; "Go";

θ_2 in X; "Go";

θ_3 in X; "Go";

θ_4 in X; "Go";

θ_5 in X; "Go";

After the last "Go" command, the program will automatically advance thru the five blocks. The program will stop with 1.1 in the Y register and the solution for X_1 in the X register. Hitting the go button brings each of the 20 values in the four solutions to the X register. The number that is in the Y register indicates which value is displayed. The integer before the decimal indicates which of the four sets of solutions is displayed and the integer after the decimal indicates which part (X_i, y_i, D_i, E_i, R_i in order) of the solution is displayed. For example, 2.3 in Y indicates that the number in the X register is D_2. A complete list is shown below:

1.1 x_1	2.1 x_2	3.1 x_3	4.1 x_4
1.2 y_1	2.2 y_2	3.2 y_3	4.2 y_4
1.3 D_1	2.3 D_2	3.3 D_3	4.3 D_4
1.4 E_1	2.4 E_2	3.4 E_3	4.4 E_4
1.5 R_1	2.5 R_2	3.5 R_3	4.5 R_4

There are three things to watch for in the use of this program. The first two involve the input data and the third involves the interpretation of the output results. First, in using this particular formulation the reference frame must be chosen such that b is never zero. (In one of the algebraic steps "a" is divided by "b" to obtain the arctan of ϕ).

Second, the input data cannot represent the special case in which one degree of freedom is constrained. When this is done, either by having all the a's, all the b's, or all the θ's with the same value in each of the five positions, then the problem is no longer a three degree of freedom problem. Mathematically, the rank of a matrix in the solution is reduced.

Third, the fourth block which solves the fourth order polynomial finds both real and imaginary roots. The fifth block operates only on the real part so there is the possibility that the results were formed from an imaginary solution and are invalid except in the case where the imaginary part is very small. Then the solution may be within engineering tolerances. Fortunately, imaginary roots to polynomials only come in pairs with identical real parts. Thus it is necessary to check for imaginary parts only when there is a double root - two sets of solutions that are identical. The imaginary parts are stored in the following locations.

Sol	Storage Location
1	0208
2	0206
3	0001
4	0003

IV. Example Problem

Given the following five desired positions for the reference axes of a moving body:

1.	$a_1 = 8.000$	$b_1 = 1.0000$	$\theta_1 = 90^\circ$
2.	$a_2 = 3.4501$	$b_2 = 1.5482$	$\theta_2 = 30^\circ$
3.	$a_3 = 1.0359$	$b_3 = 6.1340$	$\theta_3 = 30^\circ$
4.	$a_4 = 1.4645$	$b_4 = 1.9497$	$\theta_4 = 45^\circ$
5.	$a_5 = 0.0000$	$b_5 = 0.2679$	$\theta_5 = 0^\circ$

Determine the four possible (D_1, E_1) attachment points (Burmester Points) in the moving body, and for each, the corresponding radius (R) and fixed pivot (circle point) location (X,Y).

KEY	(Algebraic Symbol)	REGISTER OBSERVATION
TAPE REWIND		
TAPE READY		
LOAD PROGRAM		[illegible]
8	a_1	(8 in X)
↑	a_1 in Y	8 in Y
1	b_1	(1 in X)
CHANGE SIGN	b_1	- 1 in X
SEARCH		
0		
3.4501		
↑	a_2	3.4501 Y

KEY	(Algebraic Symbol)	REGISTER OBSERVATION
1.5482	b_2	1.5482 in X
GO		
1.0359	a_3	
↑		1.0359 in Y
6.1340	b_3	6.1340 in X
GO		
1.4645	a_4	
↑		1.4645 in Y
1.9497	$-b_4$	
CHANGE SIGN	b_4	-1.9497 in X
GO		
0		
↑	a_5	0 in Y
.2679	b_5	.2679 in X
GO		
90	θ_1	
GO		
30	θ_2	
GO		
30		
CHANGE SIGN	θ_3	
GO		
45	θ_4	
GO		
0	θ_5	
GO		Note: This step has completed the input phase.

KEY	(Algebraic Symbol)	REGISTER OBSERVATION
		1.1 IN Y
	x_1	5.470 IN X
GO		
		1.2 IN Y
	y_1	.211 IN X
GO		
		1.3 IN Y
	D_1	7.898 IN X
GO		
		1.4 IN Y
	E_1	1.738 IN X
GO		
		1.5 IN Y
	R_1	2.735 IN X
		21. IN Y
	x_2	7.750 IN X
GO		
		2.2 IN Y
	y_2	3.452 IN X
GO		
		2.3 IN Y
	D_2	7.077 IN X
GO		
		2.4 IN Y
	E_2	6.215 IN X
GO		

KEY	(Algebraic Symbol)	REGISTER OBSERVATION
		2.5 IN Y
	R_2	2.584 IN X
GO		
		3.1 IN Y
	x_3	4.0 IN X
GO		
		3.2 IN Y
	y_3	1.000 IN X
GO		
		3.3 IN Y
	D_3	5.0 IN X
GO		
		3.4 IN Y
	E_3	3.0002 IN X
GO		
		3.5 IN Y
	R_3	2.0 IN X
GO		
		4.1 IN Y
	$\bar{X}_4$	4.357 IN X
GO		
		4.2 IN Y
	$\bar{Y}_4$	2.307 IN X
GO		
		4.3 IN Y

KEY	(Algebraic Symbol)	REGISTER OBSERVATION
	D_4	3.920 IN X
GO		
		4.4 IN Y
	E_4	4.853 IN X
GO		
		4.5 IN Y
	R_4	2.320 IN X
END OF DATA		

V. Conclusions

Burmester problems of two dimensions lend themselves well to solution on the Wang 720. The language for programming is easier to learn and apply than computer languages like Fortran. The writing of a long program, once the language is known, takes slightly longer on the Wang than in Fortran. The time required to operate the program on the Wang is comparable to a demand mode remote terminal and superior to batch operation on the large digital computers.

The Wang suffers in comparison to the large computer in two major areas. Unless print-out capability is included the accuracy is limited by the ability of the operator to copy the data without error. Secondly, large programs can not be used over and over as in an iterative solution or an optimization problem. This cannot be done when the program is so large that it requires tape storage of successive blocks such as the program for the Burmester problem in two dimensions.

Nonetheless, the Wang machine is especially well suited to help the student with the study of basic kinematic synthesis. The Wang user does not have to get as involved with programming. Therefore the machine does not hamper the study.

Programming on the Wang, which uses machine language, while being tedious for operation's involving matrices and subscripted variables is straight forward and easily understood.

Also important is the fact that even beyond the learning process the Wang 720 and other machines like it have the potential to establish Burmester formulation as a practical design approach in many circumstances where the computer capability would not otherwise be available.

The Wang 720 program developed in this analysis has been stored on Cassette tape.